

MULTIVARIATE P -EULERIAN POLYNOMIALS

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ABSTRACT. The P -Eulerian polynomial counts the linear extensions of a labeled partially ordered set, P , by their number of descents. It is known that the P -Eulerian polynomials are real-rooted for various classes of posets P . The purpose of this paper is to extend these results to polynomials in several variables. To this end we study multivariate extensions of P -Eulerian polynomials and prove that for certain posets these polynomials are stable, i.e., non-vanishing whenever all variables are in the upper half-plane of the complex plane. A natural setting for our proofs is the Malvenuto-Reutenauer algebra of permutations (or the algebra of free quasi-symmetric functions). In the process we identify an algebra on Dyck paths, which to our knowledge has not been studied before.

1. INTRODUCTION

The Eulerian polynomials have been studied frequently in enumerative combinatorics, as well as in other areas since they first appeared in Euler's work [15], see [22]. The n th *Eulerian polynomial* may be defined as the generating polynomial of the *descent statistic* over the symmetric group \mathfrak{S}_n :

$$A_n(x) := \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)+1},$$

where $\text{des}(\pi) := |\{1 \leq i \leq n-1 : \pi_i > \pi_{i+1}\}|$. An important property of the Eulerian polynomials is that all their zeros are real, i.e., $A_n(x)$ splits over \mathbb{R} . This was already noted by Frobenius [16], and is not an isolated phenomenon as surprisingly many polynomials appearing in combinatorics are real-rooted, see [5, 10, 25].

Recently a theory of multivariate stable (“real-rooted”) polynomials has been developed [1, 2, 30]. A multivariate polynomial is *stable* if it is nonzero whenever all the variables have positive imaginary parts. Hence a univariate polynomial with real coefficients is stable if and only if all its zeros are real. Efforts have been made to lift results concerning the zero distribution of univariate polynomials in combinatorics to concern multivariate extensions of the polynomials, see [9, 11, 17, 18]. There are several benefits of such a refinement. Firstly the stability of the multivariate polynomial implies the real-rootedness of the univariate polynomial. Secondly, the proofs of the multivariate statements are often simpler, and may lead to a better understanding of the combinatorial setting in question. Most importantly multivariate stability implies several inequalities, refining unimodality and log-concavity, among the coefficients, see [3, 30].

An extension of the Eulerian polynomials to labeled posets was introduced in Stanley's thesis [24], and further studied in [4, 6, 7, 23, 28, 29]. We define a *labeled*

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poset on n elements to be a poset $P = ([n], \preceq)$ where \preceq is the partial order and \leq is the natural order on $[n] := \{1, 2, \dots, n\}$. The *Jordan-Hölder set* of P is the set of linear extensions of P :

$$\mathcal{L}(P) := \{\pi \in \mathfrak{S}_n : \text{if } \pi_i \preceq \pi_j \text{ then, } i \leq j \text{ for all } i, j \in [n]\},$$

where each permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ in the symmetric group, \mathfrak{S}_n , is written in one-line notation. The *P-Eulerian polynomial* is defined by

$$A_P(x) := \sum_{\pi \in \mathcal{L}(P)} x^{\text{des}(\pi)+1}. \quad (1)$$

Thus the Eulerian polynomial, $A_n(x)$, is the P -Eulerian polynomial of an n -element anti-chain, i.e., the poset on n elements with no relations.

The Neggers-Stanley conjecture asserted that for each labeled poset P , $A_P(x)$ is real-rooted, see [5, 10, 21]. The conjecture was disproved in [8], and for natural labelings it was disproved in [27]. The conjecture was proved for several classes of posets in [4, 28], and it is still open for the important class of naturally labeled graded posets.

In this paper we introduce and study a multivariate version of the P -Eulerian polynomials. We prove that these polynomials are stable for classes of labeled posets for which the univariate P -Eulerian polynomials are known to be real-rooted. In particular we prove that stability of multivariate P -Eulerian polynomials respects disjoint unions of posets (Corollary 2.3). We argue that the natural context for this is the algebra of free quasi-symmetric functions [14, 20]. In the process we identify a graded algebra, \mathcal{D} , on Dyck paths, which to our knowledge has not been studied before. One of our main theorems may be formulated as: The multiplication in \mathcal{D} preserves stability (for an appropriate notion of stability of weighted sums of Dyck paths).

The multivariate P -Eulerian polynomial is also an extension of Stembridge's peak polynomial [26, 27]. We introduce a multivariate peak polynomial for labeled posets P , and prove that it is nonzero whenever all variables are in the open right half-plane of the complex plane, whenever the multivariate P -Eulerian polynomial is stable.

2. MULTIVARIATE P -EULERIAN POLYNOMIALS

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$, let

$$\text{AB}(\pi) := \{\pi_i \in [n] : \pi_i < \pi_{i+1}\} \quad \text{and} \quad \text{DB}(\pi) := \{\pi_i \in [n] : \pi_{i-1} > \pi_i\},$$

where $\pi_0 = \pi_{n+1} := \infty$, denote the set of *ascent bottoms* and *descent bottoms* of π , respectively.

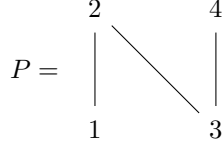
Let $[n]' := \{i' : i \in [n]\}$ be a distinct copy of $[n]$. For a permutation $\pi \in \mathfrak{S}_n$ define a monomial in the variables $\mathbf{z} = \{z_e : e \in [n] \cup [n]'\}$:

$$w_1(\pi) := \prod_{e \in \text{DB}(\pi)} z_e \prod_{e \in \text{AB}(\pi)} z_{e'} = \prod_{e \in \text{B}(\pi)} z_e,$$

where $\text{B}(\pi) = \text{DB}(\pi) \cup \{e' : e \in \text{AB}(\pi)\}$. The *multivariate P-Eulerian polynomial* is defined as

$$A_P(\mathbf{z}) := \sum_{\pi \in \mathcal{L}(P)} w_1(\pi).$$

Example 2.1. Let P be the poset below.



Then $\mathcal{L}(P) = \{1324, 1342, 3124, 3142, 3412\}$ and

$$A_P(\mathbf{z}) = z_1 z_2 z_1' z_2' z_4' + z_1 z_2 z_1' z_2' z_3' + z_1 z_3 z_1' z_2' z_4' + z_1 z_2 z_3 z_1' z_2' + z_1 z_3 z_1' z_2' z_3'.$$

Note that $A_P(\mathbf{z})$ is a polynomial in $2n$ variables, and homogeneous of degree $n + 1$. For anti-chains these polynomials were first considered by the first author in [9], where they were proven to be stable. An $(n - 1)$ -variable specialization for anti-chains was earlier defined in [17], but not proven to be stable.

Remark 2.1. Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$. For $0 \leq i \leq n$, the i th *slot* of π is the “space” between π_i and π_{i+1} , where $\pi_0 = \pi_{n+1} = \infty$. A slot is uniquely determined by an element of $B(\pi)$, namely π_i' if $\pi_i < \pi_{i+1}$ and π_{i+1} if $\pi_i > \pi_{i+1}$.

An *internal slot* of π is a slot which is not the first or the last slot of π . An internal slot is uniquely determined by an element of $([n] \cup [n']') \setminus B(\pi)$, namely π_{i+1} if $\pi_i < \pi_{i+1}$ and π_i' if $\pi_i > \pi_{i+1}$ ($1 \leq i \leq n - 1$).

We define the *disjoint union* of two labeled posets $P = ([n], \preceq_P)$ and $Q = ([m], \preceq_Q)$ on ground sets $[n]$ and $[m]$ to be the labeled poset $P \sqcup Q = ([n + m], \preceq)$ whose set of relations is

$$\{i \preceq j : i, j \in [n] \text{ and } i \preceq_P j\} \cup \{(n + i) \preceq (n + j) : i, j \in [m] \text{ and } i \preceq_Q j\}.$$

We want to see the effect on multivariate Eulerian polynomials upon taking disjoint unions. For two labeled posets P and Q with $A_P(\mathbf{z})$ and $A_Q(\mathbf{z})$ stable, we will analyze $A_{P \sqcup Q}(\mathbf{z})$ with respect to stability using free quasi-symmetric functions. In Section 4 we prove the following theorem.

Theorem 2.2. *Let P and Q be labeled posets. If $A_P(\mathbf{z})$ and $A_Q(\mathbf{z})$ are stable, then so is $A_{P \sqcup Q}(\mathbf{z})$.*

We also consider a more general definition of disjoint union. Let $P = ([n], \preceq_P)$ and $Q = ([m], \preceq_Q)$ be labeled posets, and let $S \subseteq [n + m]$ be a set of size n . Order the element of S and $T = [n + m] \setminus S$ in increasing order $s_1 < \cdots < s_n$ and $t_1 < \cdots < t_m$. Define $P \sqcup_S Q = ([n + m], \preceq)$ to be the poset on $[n + m]$ with relations $s_i \preceq s_j$ if and only if $i \preceq_P j$, and $t_i \preceq t_j$ if and only if $i \preceq_Q j$. Similarly, if P_1, \dots, P_m are labeled posets with $|P_i| = n_i$, $i \in [m]$, and $\sum_{i=1}^m n_i = n$ we may define $P_1 \sqcup_{S_1} P_2 \sqcup_{S_2} \cdots \sqcup_{S_{m-1}} P_m$ for any ordered partition $S_1 \cup \cdots \cup S_m = [n]$ with $|S_i| = n_i$, for all $i \in [m]$.

Corollary 2.3. *Let $P = ([n], \preceq_P)$ and $Q = ([m], \preceq_Q)$ be two labeled posets. If $A_P(\mathbf{z})$ and $A_Q(\mathbf{z})$ are stable, then so is $A_{P \sqcup_S Q}(\mathbf{z})$ for any $S \subseteq [n + m]$ with $|S| = n$. Indeed $A_{P \sqcup_S Q}(\mathbf{z})$ and $A_{P \sqcup Q}(\mathbf{z})$ differ only by a permutation of the variables.*

Proof. Suppose $S \neq [n]$ and let $T = [n + m] \setminus S$. Then there are $s \in S$ and $t \in T$ such that $t = s - 1$. Indeed let s be the smallest element of S such that there is a $t \in T$ such that $t < s$. Then $s - 1 \in T$. Hence let $s \in S$ and $t \in T$ be such that $t = s - 1$. Consider $\tilde{S} = S \cup \{t\} \setminus \{s\}$. For $\pi \in \mathfrak{S}_{n+m}$, let $\tilde{\pi}$ be the permutation obtained by swapping the letters s and t in π . Clearly $\pi \in \mathcal{L}(P \sqcup_S Q)$ if and only

if $\tilde{\pi} \in \mathcal{L}(P \sqcup_{\tilde{S}} Q)$. Moreover, since s and t are not related, if s and t are adjacent in π , then

$$\pi \in \mathcal{L}(P \sqcup_S Q) \Leftrightarrow \tilde{\pi} \in \mathcal{L}(P \sqcup_S Q) \Leftrightarrow \pi \in \mathcal{L}(P \sqcup_{\tilde{S}} Q) \Leftrightarrow \tilde{\pi} \in \mathcal{L}(P \sqcup_{\tilde{S}} Q).$$

If s and t are not adjacent in π , then $w_1(\tilde{\pi})$ is obtained from $w_1(\pi)$ by swapping the variables z_s and z_t , as well as the variables $z_{s'}$ and $z_{t'}$. Hence $A_{P \sqcup_{\tilde{S}} Q}(\mathbf{z})$ is obtained from $A_{P \sqcup_S Q}(\mathbf{z})$ by the same change of variables.

If $\tilde{S} = [n]$ we are done. Otherwise continue the process with $S = \tilde{S}$. This process will terminate, and then $\tilde{S} = [n]$. Indeed the sum of the elements in \tilde{S} is strictly smaller than the sum of the elements in S . \square

We argue that the natural setting for Theorem 2.2 is the algebra of free quasi-symmetric functions, see [14, 20]. Let $\text{FQSym} = \bigoplus_{n=0}^{\infty} \text{FQSym}_n$ be a \mathbb{C} -linear vector space where FQSym_n has basis $\{\pi : \pi \in \mathfrak{S}_n\}$. The (shuffle-) product on FQSym may be defined on basis elements as

$$\pi \sqcup \sigma = \sum_{\tau \in \mathcal{L}(P_\pi \sqcup P_\sigma)} \tau,$$

where P_π is the labeled chain $\pi_1 \prec \dots \prec \pi_n$. That is, the shuffle product $\pi \sqcup \sigma$ is the sum over all ways of interleaving the two permutations π and σ , where $\hat{\sigma} = (\sigma_1 + n) \dots (\sigma_m + n)$. For example

$$\begin{aligned} 132 \sqcup 21 &= 13254 + 13524 + 13542 + 15324 + 15342 \\ &+ 15432 + 51324 + 51342 + 51432 + 54132. \end{aligned}$$

Let $\mathcal{E} = \{1, 2, \dots, 1', 2', \dots\}$. Extend w_1 linearly to a weight function $w_1 : \text{FQSym} \rightarrow \mathbb{C}[z_e : e \in \mathcal{E}]$. We will now introduce two linear operators on $\mathbb{C}[z_e : e \in \mathcal{E}]$. For $e \in \mathcal{E}$, let η_e be the linear *creation operator* defined by first setting $z_e = 0$ in a polynomial and then multiplying it by z_e . Moreover, for a finite set $S \subseteq \mathcal{E}$, let

$$\eta^S := \prod_{e \in S} \eta_e,$$

where η^\emptyset is the identity operator. Let further ∂^S be the *annihilation operator*

$$\partial^S := \prod_{e \in S} \frac{\partial}{\partial z_e},$$

where ∂^\emptyset is the identity operator. We also introduce an operation, Γ_k , that shifts the variables of a polynomial as

$$\Gamma_k(f(z_1, z_2, \dots, z_{1'}, z_{2'} \dots)) = f(z_{1+k}, z_{2+k}, \dots, z_{(1+k)'}, z_{(1+k)'}', \dots).$$

Lemma 2.4. *Let $f \in \text{FQSym}_n$ and $g \in \text{FQSym}_m$, where $mn \geq 1$. Let also $\mathcal{F} = [n] \cup [n]'$ and $\mathcal{G} = [n+1, n+m] \cup [n+1, n+m]'$. Then $w_1(f \cdot g)$ only depends on $w_1(f)$ and $w_1(g)$. Moreover*

$$w_1(f \cdot g) = \Phi(w_1(f) \Gamma_n(w_1(g))),$$

where

$$\Phi = \sum_{T, S} \eta^T \partial^S,$$

and where the sum is over all $T \subseteq \mathcal{F}, S \subseteq \mathcal{G}$ for which $|S| = |T| + 1$.

Proof. It suffices to prove the lemma for basis elements. Let $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_m$. A permutation in $\mathcal{L}(P_\pi \sqcup P_\sigma)$ is uniquely determined by a subset S of the slots of σ and a subset T of the internal slots of π for which $|S| = |T| + 1$. Indeed we may factor π as $\pi = v_1 \cdots v_k$ according to T , and $\hat{\sigma} := \hat{\sigma}_0(\sigma_1 + n) \cdots (\sigma_m + n)\hat{\sigma}_{m+1}$, where $\hat{\sigma}_0 = \hat{\sigma}_{m+1} = \infty$, as $\hat{\sigma} = w_1 \cdots w_{k+1}$ according to S . Then we get a unique permutation $\tau \in \mathcal{L}(P_\pi \sqcup P_\sigma)$ for which $\tau_0 \tau_1 \cdots \tau_{n+m+1} = w_1 v_1 w_2 \cdots v_k w_{k+1}$, where $\tau_0 = \tau_{n+m+1} = \infty$.

The next step is to see how the descent and ascent bottoms of π and σ are transferred to τ .

Consider the effect of inserting v_i between w_i and w_{i+1} . Let $s \in S$ be the element of $B = B(\sigma)$ that determines the slot between w_i and w_{i+1} (see Remark 2.1). Since the letters of v_i are smaller than those of $\hat{\sigma}$ the letter s will be removed from B . This corresponds to the action of $\partial/\partial z_s$.

Consider the effect of inserting w_i between v_{i-1} and v_i for $1 < i \leq k$. Let $t \in T$ be the element of $\mathcal{F} \setminus B(\pi)$ that determines the internal slot between v_{i-1} and v_i . Since the letters of w_i are greater than those of π , the letter t will be added to B . This corresponds to action of η_t . Nothing happens when w_1 or w_{k+1} are inserted at the ends. The lemma now follows. \square

Since $w_1(fg)$ only depends on $w_1(f)$ and $w_1(g)$, Lemma 2.4 provides a “descent-ascent-bottom” algebra which is a quotient of FQSym. This algebra may thus be defined by

$$\text{DAB}_n = \text{span}_{\mathbb{C}}\{w_1(\sigma) : \sigma \in \mathfrak{S}_n\}, \text{ and } \text{DAB} = \bigoplus_{n=0}^{\infty} \text{DAB}_n,$$

with multiplication of homogeneous elements defined by $f \bullet 1 = 1 \bullet f = f$ and

$$f \bullet g = \Phi(f\Gamma_n(g)),$$

if $f \in \text{DAB}_n$, $g \in \text{DAB}_m$, where $mn \neq 0$. By Lemma 2.4, $w_1 : \text{FQSym} \rightarrow \text{DAB}$ is an algebra homomorphism. We will see in Section 3 that DAB may be viewed as an algebra of Dyck paths and that the dimension of DAB_n is the n th Catalan number $C_n = \binom{2n}{n}/(n+1)$.

3. AN ALGEBRA OF DYCK PATHS

Recall that a *Dyck path* of length $2n$ is a path in $\mathbb{N} \times \mathbb{N}$ starting from $(0, 0)$ and ending in $(2n, 0)$ using $2n$ steps, where each step is represented by one of the vectors $(1, 1)$ and $(1, -1)$. We call $u = (1, 1)$ an *up step*, and $d = (1, -1)$ a *down step*. The number of Dyck paths of length n is equal to the n th Catalan number. For us it will be convenient to code a Dyck path w_1, w_2, \dots, w_{2n} , where $w_i \in \{u, d\}$, as the word

$$v_1 v_2 \cdots v_{2n} = u w_1 \cdots w_{2n-1}. \quad (2)$$

Since w_{2n} is always a down-step we lose no information by this representation.

Define operators $\partial_1, \partial_2, \dots$ and η_1, η_2, \dots on the algebra of noncommutative polynomials, $\mathbb{C}\langle u, d \rangle$, as follows. If $v_1 v_2 \cdots v_n$ is a word with letters in $\{u, d\}$ and i is a positive integer, then

$$\partial_i(v_1 \cdots v_n) = \begin{cases} v_1 \cdots v_{i-1} d v_{i+1} \cdots v_n & \text{if } v_i = u, \\ 0 & \text{if } i > n \text{ or } v_i = d, \end{cases}$$

and dually

$$\eta_i(v_1 \cdots v_n) = \begin{cases} v_1 \cdots v_{i-1} u v_{i+1} \cdots v_n & \text{if } v_i = d, \\ 0 & \text{if } i > n \text{ or } v_i = u. \end{cases}$$

Moreover if $S \subseteq \{1, 2, \dots\}$ is a finite set, then $\partial^S = \prod_{i \in S} \partial_i$ and $\eta^S = \prod_{i \in S} \eta_i$. Endow $\mathbb{C}\langle u, d \rangle$ with a product \bullet given by $f \bullet 1 = 1 \bullet f = f$ and

$$(w_1 w_2 \cdots w_m) \bullet (v_1 v_2 \cdots v_n) = \sum_{T, S} \eta^T(w_1 \cdots w_m) \partial^S(v_1 \cdots v_n),$$

where the sum is over all finite sets $S, T \subset \{1, 2, \dots\}$ such that $|S| = |T| + 1$, whenever $mn > 0$.

Lemma 3.1. *If $w = w_1 w_2 \cdots w_{2m}, v = v_1 v_2 \cdots v_{2n} \in \mathbb{C}\langle u, d \rangle$ are Dyck paths represented as in (2), then $w \bullet v$ is a sum of words which all represent Dyck paths.*

Proof. A word $w = w_1 w_2 \cdots w_{2n}$ in the alphabet $\{u, d\}$ represents a Dyck path as in (2) if and only if $w_1 = w_2 = u$ and the corresponding path w_1, \dots, w_{2m} (in the (x, y) -plane) is a path from $(0, 0)$ to $(2m, 2)$ which crosses $y = 1$ exactly once. Clearly the path corresponding to $\eta^T(w_1 \cdots w_{2m}) \partial^S(v_1 \cdots v_{2n}) = P$ does not cross $y = 1$ when $1 < x \leq 2m$. Let $h(x)$ be the height of the path corresponding to $v_1 \cdots v_{2n}$ after x steps. If $2n < 2n + x \leq 2(n + m)$ then the height in P after $2n + x$ steps is at least $2|S| + 2 - 2|T| + h(x) \geq h(x)$, since the path corresponding to $w_1 \cdots w_m$ ends at height 2, and we have turned $|S|$ down steps to up steps and at most $|T|$ up steps to down steps. \square

By Lemma 3.1 we have a graded Dyck algebra

$$\mathcal{D} = \oplus_{n=0}^{\infty} \mathcal{D}_n,$$

where \mathcal{D}_n is the span of all Dyck paths $v_1 v_2 \cdots v_{2n}$ coded as in (2). We will now see that \mathcal{D} is isomorphic to DAB. First define an algebra homomorphism $\Theta : \text{DAB} \rightarrow (\mathbb{C}\langle u, d \rangle, \bullet)$ as follows. If M_n is a monomial defining a basis element of DAB_n , let $\Theta(M_n) = v_1 v_2 \cdots v_{2n}$ be defined as follows. For each $i \in [n]$ let

- $v_{2i-1} = u$ if and only if x_i appears in M_n , and
- $v_{2i} = u$ if and only if $x_{i'}$ appears in M_n .

By construction of the product \bullet on $\mathbb{C}\langle u, d \rangle$, we see that Θ is an algebra homomorphism.

Theorem 3.2. *The map Θ is an algebra isomorphism between the algebras DAB and \mathcal{D} .*

Proof. Clearly $\Theta(z_1 z_{1'}) = uu$ is the unique basis element of \mathcal{D}_1 . In FQSym we have the identity

$$1^n = 1 \sqcup 1 \sqcup \cdots \sqcup 1 = \sum_{\sigma \in \mathfrak{S}_n} \sigma,$$

where $1 \in \mathfrak{S}_1$. Since $w_1 : \text{FQSym} \rightarrow \text{DAB}$ and $\Theta : \text{DAB} \rightarrow (\mathbb{C}\langle u, d \rangle, \bullet)$ are algebra homomorphisms, with $w_1(1) = z_1 z_{1'}$, we see that $\Theta(\text{DAB}_n)$ is the span of all words in the support of

$$(uu)^n = (uu) \bullet \cdots \bullet (uu) \in \mathcal{D}_n.$$

Hence $\Theta : \text{DAB} \rightarrow \mathcal{D}$ by Lemma 3.1. The homomorphism $\Theta : \text{DAB} \rightarrow \mathcal{D}$ is injective since $\Theta : \text{DAB} \rightarrow (\mathbb{C}\langle u, d \rangle, \bullet)$ is injective. To prove surjectivity it remains to prove that for any Dyck path $v \in \mathcal{D}_n$ there is a Dyck path $w \in \mathcal{D}_{n-1}$ such that v is in the

support of $(uu) \bullet w$. Let w be the word obtained by first changing the first down step (say at position j) in v to an up set and then deleting the first two letters. Then w is a Dyck path and $v = uu\partial_{j-2}(w)$ is in the support of $(uu) \bullet w$. \square

Example 3.1. The product of $uudu, uuud \in \mathcal{D}_2$ is $uudu \bullet uuud = uudu uudd + uu ududd + uu uuudd + u udu duud + uu udu dud + u udu vudud$.

Remark 3.3. There is a much studied graded algebra on rooted planar binary trees called the Loday-Ronco algebra [19]. The Loday-Ronco algebra is a sub-algebra of FQSym , and rooted planar binary trees are in bijection with Dyck paths. Hence it is natural to ask if this algebra and \mathcal{D} are isomorphic. We have not found such an isomorphism.

4. PRODUCTS PRESERVING STABILITY

To prove that Φ in Theorem 2.4 preserves stability we need two theorems on stable polynomials. The first theorem is a version of the celebrated Grace-Walsh-Szegő theorem, see e.g. [1, Proposition 3.4].

Let $\Omega \subset \mathbb{C}^n$. A polynomial $P(\mathbf{z}) \in \mathbb{C}[z_1, \dots, z_n]$ is Ω -stable if

$$\mathbf{z} \in \Omega \quad \text{implies} \quad P(\mathbf{z}) \neq 0.$$

Theorem 4.1. Let $P(z_1, z_2, \dots, z_n)$ be a polynomial, let $H \subset \mathbb{C}$ be an open half-plane, and $1 \leq k \leq n$. If P is of degree at most one in z_i for each $1 \leq i \leq k$ and symmetric in the variables z_1, \dots, z_k , then $P(z_1, z_2, \dots, z_n)$ is H^n -stable if and only if the polynomial $P(z_1, z_1, \dots, z_1, z_{k+1}, \dots, z_n)$ is H^{n-k+1} -stable.

The next theorem is a special case of a recent characterization of stability preservers in [1]. Let $\mathbb{C}_1[\mathbf{z}] = \mathbb{C}_1[z_1, \dots, z_n]$ be the space of polynomials of degree at most one in z_i for all i . For a linear operator $T : \mathbb{C}_1[\mathbf{z}] \rightarrow \mathbb{C}_1[\mathbf{z}]$ define its *symbol* by

$$G_T(\mathbf{z}, \mathbf{w}) = T[(z_1 + w_1) \cdots (z_n + w_n)] = \sum_{S \subseteq [n]} T(\mathbf{z}^S) \mathbf{w}^{[n] \setminus S}.$$

Theorem 4.2 ([1]). Let $\Omega = \{z \in \mathbb{C} : \text{Im}(z) > 0\}^n$ or $\Omega = \{z \in \mathbb{C} : \text{Re}(z) > 0\}^n$ and let $T : \mathbb{C}_1[\mathbf{z}] \rightarrow \mathbb{C}_1[\mathbf{z}]$ be a linear operator.

- If $G_T(\mathbf{z}, \mathbf{w})$ is $\Omega \times \Omega$ -stable, then T preserves Ω -stability.
- If the rank of T is greater than one and T preserves Ω -stability, then $G_T(\mathbf{z}, \mathbf{w})$ is $\Omega \times \Omega$ -stable.

Lemma 4.3. Let $m \geq 2$ and $n \geq 1$ be integers. All zeros of the polynomial

$$\sum_{k=0}^n \binom{n}{k} \binom{m}{k+1} x^k$$

are real and negative.

Proof. The lemma is a consequence of Malo's theorem (see e.g. [13, Theorem 2.4]) which asserts that if $f = \sum_{k \geq 0} a_k x^k$ is a real-rooted polynomial and $g = \sum_{k \geq 0} b_k x^k$ is a real-rooted polynomial whose zeros all have the same sign, then the polynomial

$$f * g = \sum_{k \geq 0} a_k b_k x^k$$

is real-rooted. Indeed, the polynomial in the statement of the theorem is

$$x^{-1}(x(x+1)^n * (x+1)^m).$$

□

We are now ready to state and prove the main theorem of this section. Note that Theorem 2.2 immediately follows from the theorem below.

Theorem 4.4. *Let $f, g \in FQSym$ be two homogeneous elements and let w_1 be defined as before. If $w_1(f)$ and $w_1(g)$ are stable, then so is $w_1(f \cdot g)$.*

Proof. Note that a homogeneous polynomial is H^n -stable for an open half-plane H with boundary containing the origin if and only if it is J^n -stable for some (and then each) open half-plane J with boundary containing the origin. Hence it suffices to prove that Φ preserves stability with respect to the open right half-plane (Hurwitz stability). Recall that Φ acts on multi-affine polynomials in the variables $\{z_e : e \in \mathcal{F} \cup \mathcal{G}\}$ where \mathcal{F} and \mathcal{G} are disjoint sets. Now

$$\begin{aligned} \partial^S \prod_{e \in \mathcal{G}} (z_e + w_e) &= \prod_{e \in \mathcal{G} \setminus S} (z_e + w_e) = \prod_{e \in \mathcal{G}} (z_e + w_e) \prod_{e \in S} (z_e + w_e)^{-1} \quad \text{and} \\ \eta^T \prod_{f \in \mathcal{F}} (z_f + w_f) &= \prod_{f \in T} z_f w_f \prod_{f \in \mathcal{F} \setminus T} (z_f + w_f) = \prod_{f \in \mathcal{F}} (z_f + w_f) \prod_{f \in T} z_f w_f (z_f + w_f)^{-1}. \end{aligned}$$

Hence we may write the symbol of Φ as

$$G_\Phi(\mathbf{z}, \mathbf{w}) = \prod_{e \in \mathcal{G}} (z_e + w_e) \prod_{f \in \mathcal{F}} (z_f + w_f) \sum_{S, T} \prod_{e \in S} y_e \prod_{f \in T} x_f,$$

where the sum is over all $S \subseteq \mathcal{G}, T \subseteq \mathcal{F}$ for which $|S| = |T| + 1$,

$$x_f = z_f w_f (z_f + w_f)^{-1} = \left(\frac{1}{w_f} + \frac{1}{z_f} \right)^{-1},$$

and $y_e = (z_e + w_e)^{-1}$ for all $f \in \mathcal{F}$ and $e \in \mathcal{G}$. Since the open right half-plane is invariant under $z \mapsto z^{-1}$ it suffices to prove that the polynomial

$$\sum_{S, T} \prod_{e \in S} y_e \prod_{f \in T} x_f$$

is Hurwitz stable. This polynomial is symmetric in \mathbf{x} and in \mathbf{y} , so by Theorem 4.2 it remains to prove that the polynomial

$$y \sum_{k=0}^n \binom{n}{k} \binom{m}{k+1} x^k y^k =: yP(xy)$$

is Hurwitz stable. The polynomial $P(x)$ has only real and negative zeros by Lemma 4.3, so that $P(xy)$ is a product of factors of the form $a + xy$ where $a > 0$. The product of two numbers in the open right half-plane is never a negative real number, from which the proof follows. □

We will now generalize Theorem 4.4 to other weights. Define weight functions $w_j : \text{FQSym} \rightarrow \mathbb{C}[x, y, z_1, z_2, \dots]$ for $2 \leq j \leq 4$ by

$$\begin{aligned} w_2(\pi) &= y^{|\pi| - \text{des}(\pi)} \prod_{i \in \text{DB}(\pi)} z_i, \\ w_3(\pi) &= y^{\text{des}(\pi) + 1} \prod_{i \in \text{AB}(\pi)} z_i \text{ and} \\ w_4(\pi) &= x^{\text{des}(\pi) + 1} y^{|\pi| - \text{des}(\pi)}. \end{aligned}$$

For a finite set A of indices, let S_A be the symmetrization with respect to the variables indexed by A , that is,

$$S_A = \frac{1}{|A|!} \sum_{\pi \in \mathfrak{S}(A)} \pi,$$

where π acts on the variables of the polynomial by permuting them according to π . For a proof of the next lemma see e.g. Theorem 1.2 and Proposition 1.5 in [2].

Lemma 4.5. *Let $A \subseteq [n]$ and suppose $f \in \mathbb{C}[z_1, \dots, z_n]$ is stable and has degree at most one in z_i for each $i \in A$. Then $S_A(f)$ is stable.*

Lemma 4.6. *Let $\{z_i\}_{i \in F}$ and $\{z_i\}_{i \in G}$ be disjoint set of variables and suppose that f and g are multi-affine polynomials that depend only on the variables indexed by F and G , respectively. Let*

$$\Phi = \sum_{S, T} \eta^T \partial^S,$$

where the sum is over all $S \subseteq F$, $T \subseteq G$ for which $|S| = |T| + 1$.

If $A \subseteq F$ and $B \subseteq G$, then

$$S_A S_B \Phi(fg) = \Phi(S_A f \cdot S_B g).$$

Proof. The lemma follows since Φ acts symmetrically on the variables in F and G , and

$$\pi(\partial^S g) = \partial^{\pi(S)} \pi(g) \quad \text{and} \quad \pi(\eta^T f) = \eta^{\pi(T)} \pi(f),$$

where $\pi(S) = \{\pi(s) : s \in S\}$. \square

In Lemma 2.4 we proved that the weight w_1 of a product of two elements in FQSym only depend on the weights (w_1) of the two elements. In the next lemma we prove the same statement for the weights w_2, w_3 and w_4 .

Lemma 4.7. *Let $f, g \in \text{FQSym}$ and $1 \leq i \leq 4$. Then $w_i(f \cdot g)$ only depends on $w_i(f)$ and $w_i(g)$.*

Moreover if $w_i(f)$ and $w_i(g)$ are homogenous and stable, then so is $w_i(f \cdot g)$.

Proof. We prove the lemma for $i = 4$. The other cases follow similarly. Let $f \in \text{FQSym}_n$ and $g \in \text{FQSym}_m$. For $1 \leq i \leq j$ let $[i, j] = \{i, i+1, \dots, j\}$ and $[i, j]' = \{i', \dots, j'\}$. Note that $w_4(f \cdot g)$ carries precisely the same information as $S_{[1, n+m]} S_{[1, n+m]'} w_1(f \cdot g)$. Now, by applying Lemma 4.6

$$\begin{aligned} S_{[1, n+m]} S_{[1, n+m]'} w_1(f \cdot g) &= S_{[1, n+m]} S_{[1, n+m]'} \Phi(w_1(f) \Gamma_n(w_1(g))) \\ &= S_{[1, n+m]} S_{[1, n+m]'} S_{[1, n]} S_{[1, n]'} S_{[n+1, n+m]} S_{[n+1, n+m]'} \Phi(w_1(f) \Gamma_n(w_1(g))) \\ &= S_{[1, n+m]} S_{[1, n+m]'} \Phi(FG), \end{aligned}$$

where $F = S_{[1,n]}S_{[1,n]}'w_1(f)$ and $G = S_{[n+1,n+m]}S_{[n+1,n+m]}'\Gamma_n(w_1(g))$. This proves the first statement.

Now, F is stable if and only if $w_4(f)$ is stable by Theorem 4.1. This completes the proof by Theorem 4.4 and Lemma 4.5. \square

5. APPLICATIONS

For a labeled poset P we define a *descent bottom P -Eulerian polynomial* by

$$A_P^{\text{DB}}(\mathbf{z}) = \sum_{\pi \in \mathcal{L}(P)} \prod_{e \in \text{DB}(\pi)} z_e.$$

Corollary 5.1. *Let P and Q be labeled posets on $[n]$ and $[m]$, respectively. Suppose $A_P^{\text{DB}}(\mathbf{z})$ and $A_Q^{\text{DB}}(\mathbf{z})$ are stable. If $S \subset [n+m]$ is an n -set, then the polynomial $A_{P \sqcup_S Q}^{\text{DB}}(\mathbf{z})$ is stable.*

Proof. As in the proof of Corollary 2.3 it suffices to consider the case when $S = [n]$. Suppose $A_P^{\text{DB}}(\mathbf{z})$ and $A_Q^{\text{DB}}(\mathbf{z})$ are stable. Then so are the homogenized polynomials

$$\sum_{\pi \in \mathcal{L}(P)} y^{n-\text{des}(\pi)} \prod_{e \in \text{DB}(\pi)} z_e \quad \text{and} \quad \sum_{\pi \in \mathcal{L}(Q)} y^{m-\text{des}(\pi)} \prod_{e \in \text{DB}(\pi)} z_e$$

by [3, Theorem 4.5]. The corollary now follows from Lemma 4.7. \square

Brenti [4] conjectured that if the univariate P -Eulerian polynomials $A_P(x)$ and $A_Q(x)$ of two labeled posets P and Q are real-rooted, then so is $A_{P \sqcup Q}(x)$. The conjecture was proved by Wagner in [28, 29]. We may now deduce it as an immediate corollary of Lemma 4.7.

Corollary 5.2. *If P and Q are labeled posets such that $A_P(x)$ and $A_Q(x)$ are real-rooted, then so is $A_{P \sqcup Q}(x)$.*

Proof. The proof follows as the proof of Corollary 5.1, using w_4 instead of w_2 . \square

Stembridge [26, 27] studied the *peak polynomial* associated to a labeled poset P . A peak in a permutation $\pi \in \mathfrak{S}_n$ is an index $1 < i < n$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. Let $\Lambda(\pi)$ be the set of peaks of π and define

$$\bar{A}_P(x) = \sum_{\pi \in \mathcal{L}(P)} x^{|\Lambda(\pi)|}.$$

Let us now define a *multivariate peak polynomial*. For $\pi \in \mathfrak{S}_n$ let

$$N(\pi) = \{\pi_i : \pi_{i-1} < \pi_i > \pi_{i+1}, i \in [n]\} \cup \{\pi_i : \pi_{i-1} > \pi_i < \pi_{i+1}, i \in [n]\},$$

where $\pi_0 = \pi_{n+1} = \infty$, be the *peak-valley set* of π and define a multivariate peak polynomial by

$$\bar{A}_P(\mathbf{z}) := \sum_{\pi \in \mathcal{L}(P)} \prod_{e \in N(\pi)} z_e.$$

Note that $|N(\pi)| = 2|\Lambda(\pi)| + 1$ so that

$$\bar{A}_P(x, x, \dots) = x \bar{A}_P(x^2).$$

Recall that a polynomial $P(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is said to be *Hurwitz stable* if $P(\mathbf{z}) \neq 0$ for all $\mathbf{z} \in \mathbb{C}^n$ with $\text{Re}(z_i) \geq 0$, for all $1 \leq i \leq n$.

Proposition 5.3. *Let P be a labeled poset. If $A_P(\mathbf{z})$ is stable, then $\bar{A}_P(\mathbf{z})$ is Hurwitz stable and $\bar{A}_P(x)$ is real-rooted.*

Proof. If $A_P(\mathbf{z})$ is stable, then $A_P(\mathbf{z})$ is Hurwitz stable by homogeneity. Set $z_{e'} = z_e$ for all $e \in E$ in $A_P(\mathbf{z})$ and denote the resulting polynomial by $H(\mathbf{z})$. Then

$$\bar{A}_P(\mathbf{z}) = \Psi(H(\mathbf{z})),$$

where Ψ is the linear operator that maps a monomial $\prod_{e \in \mathcal{E}} z_e^{k_e}$ to $\prod_{e \in \mathcal{E}} z_e^{k_e \bmod 2}$, where $k_e \bmod 2 = 0$ if k_e is even, and $k_e \bmod 2 = 1$ if k_e is odd. The polynomial $\bar{A}_P(\mathbf{z})$ is Hurwitz stable since Ψ preserves Hurwitz stability, see [12, Proposition 4.19]. Since $\bar{A}_P(\mathbf{z})$ is Hurwitz stable we may rotate the variables and deduce that

$$\bar{A}_P(ix, ix, \dots) = ix \bar{A}_P(-x^2), \quad (\text{where } i = \sqrt{-1}),$$

is stable. Hence $\bar{A}_P(x)$ is real-rooted. \square

An natural question, which is not addressed in this paper, is whether $\bar{A}_{P \sqcup Q}$ is Hurwitz stable whenever \bar{A}_P and \bar{A}_Q are Hurwitz stable, for any two labeled posets P and Q .

Let $P = ([n], \preceq_P)$ and $Q = ([m], \preceq_Q)$ be two labeled posets. We define the *ordinal sum* of P and Q to be the labeled poset $P \oplus Q = ([n+m], \preceq)$ with the following set of relations:

$$\begin{aligned} & \{i \preceq j : i, j \in [n] \text{ and } i \preceq_P j\} \cup \\ & \{(n+i) \preceq (n+j) : i, j \in [m] \text{ and } i \preceq_Q j\} \cup \\ & \{i \preceq (n+j) : i \in [n], j \in [m]\}. \end{aligned}$$

In Section 2 we saw the effect on multivariate Eulerian polynomials upon taking disjoint unions. Now we will study the effect for ordinal sums. For a labeled poset $P = ([n], \preceq_P)$ define $P_0 = ([n+1], \preceq_{P_0})$ to be the poset where $1 \preceq_{P_0} j$ for $2 \leq j \leq n+1$ and $i \preceq_{P_0} j$ if $i-1 \preceq_P j-1$ for $i, j \in \{2, \dots, n+1\}$.

Lemma 5.4. *Let $P = ([n], \preceq_P)$ be a poset such that $A_P(\mathbf{z})$ is stable and let Q be a poset such that $A_{Q_0}(\mathbf{z})$ is stable. Then $A_{P \oplus Q}(\mathbf{z})$ is stable.*

Proof. Clearly

$$\begin{aligned} \mathcal{L}(P \oplus Q) &= \{\pi_1 \cdots \pi_n (\sigma_1 + n) \cdots (\sigma_m + n) : \pi_1 \cdots \pi_n \in \mathcal{L}(P), \sigma_1 \cdots \sigma_m \in \mathcal{L}(Q)\} \\ \text{and } \mathcal{L}(Q_0) &= \{1(\sigma_1 + 1) \cdots (\sigma_m + 1) : \sigma_1 \cdots \sigma_m \in \mathcal{L}(Q)\} \end{aligned}$$

from which

$$A_{P \oplus Q}(\mathbf{z}) = \frac{A_P(\mathbf{z}) \Gamma_n(A_{Q_0}(\mathbf{z}))}{z_{n+1} z_{(n+1)'}}$$

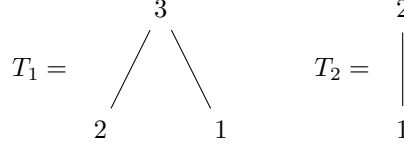
follows. Thus $A_{P \oplus Q}(\mathbf{z})$ is stable. \square

Define a *naturally labeled decreasing tree*, T , recursively as follows.

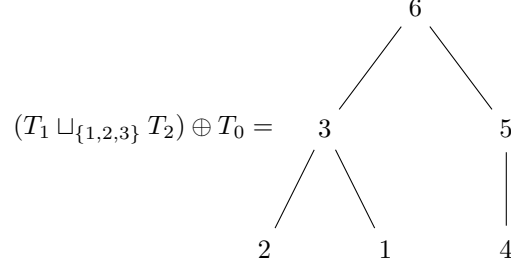
- 1) Either $T = T_0 := (\{1\}, \emptyset)$, the antichain on one element, or
- 2) $T = (T_1 \sqcup_{S_1} T_2 \sqcup_{S_2} \cdots \sqcup_{S_{m-1}} T_m) \oplus T_0$, for some ordered partition $S_1 \cup \cdots \cup S_m = [n]$, where T_i is a naturally labeled decreasing tree for all $i \in [m]$.

That is, a naturally labeled decreasing tree is a labeled poset whose Hasse diagram is a decreasing tree with the root at the top.

Example 5.1. Let T_1 and T_2 be the naturally labeled decreasing trees below.



Then



A naturally labeled decreasing forest is a disjoint union

$$F = T_1 \sqcup_{S_1} T_2 \sqcup_{S_2} \cdots \sqcup_{S_{k-1}} T_k$$

of naturally labeled decreasing trees.

Corollary 5.5. *If F is a naturally labeled decreasing forest, then $A_F(\mathbf{z})$ is stable.*

Proof. The operations defining naturally labeled decreasing trees and forests preserve stability by Corollary 2.3 and Lemma 5.4. Hence the corollary follows by induction on the size of F . \square

The *dual* of a labeled poset $P = ([n], \preceq)$ is the poset $P^* = ([n], \preceq^*)$, where $a \preceq^* b$ if and only if $b \preceq a$.

Proposition 5.6. *If $P = ([n], \preceq)$ is a poset such that $A_P(\mathbf{z})$ is stable, then $A_{P^*}(\mathbf{z})$ is stable. In fact, $A_{P^*}(z_1, \dots, z_n, z_{1'}, \dots, z_{n'}) = A_P(z_{1'}, \dots, z_{n'}, z_1, \dots, z_n)$.*

Proof. First note that $\pi^* = \pi_n \cdots \pi_1 \in \mathcal{L}(P^*)$ if and only if $\pi = \pi_1 \cdots \pi_n \in \mathcal{L}(P)$. Hence $\text{DB}(\pi) = \text{AB}(\pi^*)$ and $\text{AB}(\pi) = \text{DB}(\pi^*)$, and the proposition follows. \square

Corollary 5.7. *If F is the dual of a naturally labeled decreasing forest, then $A_F(\mathbf{z})$ is stable.*

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